# HOMEOMORPHISMS OF KNASTER CONTINUA

By

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A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
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I dedicate this to my family and to Josephine.

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In this thesis we investigate homeomorphisms of Knaster continua. We determine the minimum number of fixed-points homeomorphisms of these continua must have. This analysis is related to a question raised by William S. Mahavier on whether a homeomorphism on the Knaster bucket handle must have at least two fixed points. It is proved that an isotopy between homeomorphisms of the Knaster continuum can be lifted to an isotopy between homeomorphisms of the solenoid. We give necessary and sufficient conditions for a homeomorphism of the Knaster continuum to have at least two fixed points. We construct a Knaster continuum on which every homeomorphism has either uncountably many fixed points or uncountably many points of period 2. We determine the minimum number of fixed points a homeomorphism on the Knaster continuum can have. We construct an example to show that Bowen's theorem on entropy of quotients on compact spaces does not readily generalize to non-compact spaces.

We generalize the definitions of Knaster continua to constructions via toral homomorphisms. We show that homeomorphism on these continua (in the odd dimension case) lift to homeomorphisms to the solenoid and end with some questions for further research.

# CHAPTER 1
# INTRODUCTION

In this chapter we establish notation, definitions and some basic results from continuum theory, topological group theory and cohomology theory to be used in subsequent chapters. We assume the reader is familiar with the standard results and terminology of general topology, such as is covered in Munkres [22]. Specifically such results as the Baire Category Theorem are assumed. For less well known results such as those from algebraic topology and topological group theory the appropriate references will be cited. Basic definitions in algebraic topology such as of homotopy groups, homology groups and cohomology groups may be found in Spanier [27]. For basic definitions from topological group theory, the reader is referred to the text by Hewitt and Ross [13].

By a *map* we mean a continuous function.

## 1.1 Continua

A *continuum* is a compact connected metric space. A *subcontinuum* $Y$ of the continuum $X$ is a closed, connected subset of $X$. A continuum $X$ is *decomposable* if there exist two nonempty subcontinua $H$ and $K$ of the continuum $X$ such that $H \neq X$ and $K \neq X$, but $H \cup K = X$. Any continuum that is not decomposable is said to be *indecomposable*.

## 1.2 Composants

A *composant* $\mathrm{Com}(x)$ of a given point $x \in X$ is the union of all proper subcontinua in $X$ that contain the point $x$. A point $y$ belongs to $\mathrm{Com}(x)$ if there is a proper subcontinuum $A$ that contains both $x$ and $y$. It is known [16] that continuum

$X$ is indecomposable if and only if $\{\mathrm{Com}(x) \,|x \in X\}$ forms a partition of $X$ into an uncountable collection of first category, connected sets each of which is dense in $X$. A set is *first category* in $X$ if it can be written as the union of a countable number of nowhere dense subsets of $X$. It is known [16] that a continuum is indecomposable and nondegenerate if and only if it possesses two disjoint composants . A continuum $X$ is hereditarily indecomposable if every subcontinuum of $X$ is indecomposable.

## 1.3 Homogeneity

A continuum $X$ is said to be homogeneous if for any given points $x, y \in X$ there is a homeomorphism $h : X \longrightarrow X$ of $X$ onto $X$ such that $h(x) = y$.

## 1.4 Examples of Indecomposable Continua

The solenoids and Knaster continua (Chapter 3) are examples of indecomposable continua. The pseudoarc [16, figure 4] is an example of a heriditarily indecomposable continuum. In fact every subcontinuum of the pseudoarc is homeomorphic to the pseudoarc. Such a continuum is said to be hereditarily equivalent. The unit interval is another example of a hereditarily equivalent continuum.

## 1.5 Inverse Limit Spaces of Continua.

An inverse sequence is a double sequence $\{X_i, f_i\}_{i=1}^{\infty}$ of topological spaces $X_i$ and maps $f_i$ such that each $X_i$ is a topological space for each $i$ and each map $f_i$ takes $X_{i+1}$ to $X_i$. The collection of maps $f_i$ are referred to as *bonding maps.* We write

$$X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} X_3 \ldots.$$

The *inverse limit* of the inverse sequence is the set

$$X_\infty = \varprojlim \{X_i, f_i\} = \left\{(x_1, x_2, \ldots) \in \prod_{i=1}^{\infty} X_i : \text{ for all } i \geq 1, \; f_i(x_{i+1}) = x_i\right\}$$

topologized with the relativized product topology. Let $\pi_k$ denote the *natural projection* from both $\prod_{i=1}^{\infty} X_i$ and its subset $X_\infty$ onto $X_k$ defined by $\pi_k((x_n)) = x_k$. A

basis for a topology of $X$ is a collection $\beta$ of open subsets of $X$ having the property that if $U$ is an open subset of $X$ and $p \in U$, then there exists $B \in \beta$ such that $p \in B \subset U$. The *product topology* on $\prod_{i=1}^{\infty} X_i$ is typically defined in terms of a basis which consists of all sets of the form

$$\left\{\pi_{n_1}^{-1}(U_{n_1}) \cap \cdots \cap \pi_{n_k}^{-1}(U_{n_k}) : k \geq 1, \text{ and each } U_{n_i} \text{ is open in } X_{n_i}\right\}.$$

This lemma provides a significant simplification of the topology of $X_\infty$. We now exhibit a *basis* for the topology of $X_\infty$.

**Lemma 1.5.1** *The collection* $\left\{\pi_{n_k}^{-1}(U_{n_k}) : k \geq 1 \text{ and } U_k \text{ is open in } X_k\right\}$ *is a basis for the topology on* $X_\infty$.

If all the spaces $X_i$ are continua then so is the inverse limit space which would have the metric $\bar{d}$ given by

$$\bar{d}(\bar{x}, \bar{y}) = \sum_{i=0}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$$

where for each $i$, $d_i$ is a metric for $X_i$ bounded by 1. If $X_i = X$ and $f_i = f$ for all i (i.e., if there is a single bonding map involved) the inverse limit space is denoted by $(X, f)$ and the map $\hat{f} : (X, f) \to (X, f)$ defined by $\hat{f}((x_0, x_1, \ldots)) = (f(x_0), x_0, x_1, \ldots)$ is called the induced homeomorphism.

## 1.6 Arclike Continua

A continuum is said to be *arclike* if it is the inverse limit of arcs (spaces homeomorphic to the closed unit interval). Arclike continua are also referred to as chainable continua since a nondegenerate continuum $X$ is arclike if and only if it is chainable [23, lemma 12.11]. A map between metric spaces is called an $\epsilon$-*map* if the diameters of the point inverses are less than $\epsilon$. A continuum $X$ is *chainable* if for every $\epsilon > 0$ there exists an $\epsilon$-map $g : X \to [0, 1]$. A subcontinuum $Y$ of an arclike continuum $X$ is said to be an end-continuum if whenever $Y \subseteq A \cap B$ with $A$ and

$B$ proper subcontinua of $X$ then either $A \subseteq B$ or $B \subseteq A$. A point $p \in X$ is called an *end point* of $X$ if $p$ is an end-continuum in $X$. Homeomorphic continua have the same number of end-points

**Proposition 1.6.1** *Let $X$ and $Y$ be homeomorphic arclike continua. Then $X$ has the same number of end points as $Y$.*

**Proof:** Let $p \in X$ be an end point in $X$ and suppose $h : X \longrightarrow Y$ is a homeomorphism. Suppose $h(p) \in A \cap B$ for $A$ and $B$ proper subcontinua of $Y$. Then $p \in h^{-1}(A) \cap h^{-1}(B)$, $h^{-1}(A)$ and $h^{-1}(B)$ are proper subcontinua of $X$ since $h$ is a homeomorphism. Since $p$ is an end point of $X$ either $h^{-1}(A) \subseteq h^{-1}(B)$ or $h^{-1}(B) \subseteq h^{-1}(A)$. It follows that either $A \subseteq B$ or $B \subseteq A$ and therefore that $p$ $h(p)$ is an endpoint of $Y$. ■

## 1.7 Induced Maps between Inverse Limits.

**Proposition 1.7.1** *Let $\{X_n, f_n\}_{n=1}^{\infty}$ and $\{Y_n, g_n\}_{n=1}^{\infty}$ be inverse sequences, and for each $n \geq 1$ let $h_n : X_n \longrightarrow Y_n$ be a function such that $h_n f_n = g_n h_{n+1}$. Then there is an induced function $h_\infty : X_\infty \longrightarrow Y_\infty$ defined by $h_\infty((x_n)) = (h_n(x_n))$.*

If $f : X \to X$ is a mapping, a point $x$ of $X$ is called a *periodic point* for $f$ provided $f^n(x) = x$ for some positive integer $n$. A point $x$ is said to have period $k$ provided $k$ is the least positive integer such that $f^k(x) = x$. A point $x$ is a fixed point for the map $f$ if $f(x) = x$.

A continuum is said to have the fixed point property if every mapping $g : X \to X$ has at least one fixed point. Arclike continua are known to have the fixed point property [12]. A continuum is *treelike* if it can be expressed as the inverse limit of trees. Treelike continua in general do not have the fixed point property [5].

## 1.8 Lifting Maps to Covers.

We review an important problem of algebraic topology, called the lifting problem. Let $p : E \longrightarrow B$ and $f : X \longrightarrow B$ be maps. Let $X$, $B$ and $E$ be topological spaces. The *lifting problem* for $f$ is to determine whether there is a continuous map $f' : X \longrightarrow E$ such that $f = p \circ f'$, that is, whether the diagonal arrow in the diagram

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dotted] \arrow[r, "f"] & B \end{tikzcd}$$

corresponds to a continuous map making the diagram commutative. If such a map $f'$ exists, then $f$ can be lifted to $E$, and $f'$ is called a *lifting*, or *lift*, of $f$.

## 1.9 Homotopy Lifting

A map $p : E \longrightarrow B$ is said to have the *homotopy lifting property* with respect to a space $X$ if, given maps $f' : X \longrightarrow E$ and $F : X \times I \longrightarrow B$ such that $F(x, 0) = pf'(x)$ for $x \in X$, then there is a map $F' : X \times I \longrightarrow E$ such that $F'(x, 0) = f'(x)$ for $x \in X$ and $p \circ F' = F$, in other words, the diagonal map in the diagram

$$\begin{tikzcd} X \times 0 \arrow[r, "f'"] \arrow[d, "\cap"] & E \arrow[d, "p"] \\ X \times I \arrow[ur, dotted] \arrow[r, "F"'] & B \end{tikzcd}$$

exists to make the diagram commutative. A map $p : E \longrightarrow B$ is called a fibration if $p$ has the homotopy lifting property with respect to every space.

## 1.10 Covering Projection

Let $p : \tilde{X} \longrightarrow X$ be a continuous map. An open subset $U \subset X$ is said to be *evenly covered* by $p$ if $p^{-1}(U)$ is the disjoint union of open subsets of $X$ each of which is mapped homeomorphically onto $U$ by $p$. A map $p : \tilde{X} \longrightarrow X$ is called a *covering projection* if each point $x \in X$ has an open neighborhood evenly covered by $p$ [27].

**Lemma 1.10.1** *A covering projection is a fibration.*

A map $p : E \longrightarrow B$ is said to have *the unique path lifting property* if, given paths $\omega$ and $\omega\prime$ in $E$ such that $p \circ \omega = p \circ \omega'$ and $\omega(0) = \omega'(0)$, then $\omega = \omega'$. A covering projection has the unique path lifting property [27].

## 1.11 Topological Entropy.

Let $(X, d)$ be a metric space and $T : X \to X$ be a uniformly continuous map. For subsets $F, K \subset X$, say that $F$ $(n, \epsilon)$-spans $K$ (with respect to T) provided that for each $x \in K$ there is a $y \in F$ for which $d(T^j(x), T^j(y)) \le \epsilon$ for all $0 \le j \le n$.

For a compact set $K \subset X$ let $r_n(\epsilon, K)$ be the smallest cardinality of any set $F$ which $(n, \epsilon)$-spans $K$ (with respect to $T$). One can interpret the meaning of that quantity as the minimal number of initial conditions whose behavior up to time n approximates the behavior of any initial condition up to $\epsilon$. Let

$$\bar{r}_T(\epsilon, K) := \limsup_{n\to\infty} \frac{1}{n} \log\left(r_n(\epsilon, K)\right)$$

Thus $\bar{r}_T(\epsilon, K)$ measures the exponential growth rate for the quantity $r_n(\epsilon, K)$. The proof of the following lemma is in Bowen [8].

**Lemma 1.11.1** *Let $(X, d)$ be a metric space. Let $T : X \to X$ be a uniformly continuous map. Let $\epsilon > 0$. For each compact set $K \subset X$ and each positive integer $n$, the quantity $r_n(\epsilon, K) < \infty$.*

For a uniformly continuous map T on X and $K \subset X$ compact, set

$$h_d(T, K) := \lim_{\epsilon\to 0} \bar{r}_T(\epsilon, K)$$

and

$$h_d(T) := \sup_{K \text{ compact}} h_d(T, K).$$

One of the results in this dissertation is determining the minimum number of periodic points that homeomorphisms of certain arclike continua can have. The continua studied in some detail are inverse limit spaces of the unit interval $[-1, 1]$ with the Chebychev polynomials (3.4) as bonding maps. These spaces are quotients of a certain class of topological groups. We need some definitions and results from topological group theory and algebraic topology. Some of these results are in the next chapter.

## 1.12 Topological Groups

A topological group is a Hausdorff topological space $X$ that is also a group such that the following conditions hold:

1. The multiplication $m : X \times X \to X$ is a continuous operation where$X \times X$ is equipped with the product topology.

2. The inversion operation $x \mapsto x^{-1} : X \to X$ is continuous.

A *topological group homomorphism* is a map $g : X \to Y$ between the topological groups $X$ and $Y$ that is also a homomorphism. The unit circle $S^1$ and the tori $T^n$, with n an integer, are examples of abelian topological groups.

Consider a sequence $(X_i)_{i=1}^{\infty}$ of topological groups that are also continua, along with topological group homomorphisms $f_i : X_{i+1} \to X_i$. Then the inverse limit of $\{X_i, f_i\}_{i=1}^{\infty}$ is a topological subgroup of $\prod_{i=1}^{\infty} X_i$. We shall define the solenoids as inverse limits of the circle under certain bonding maps.

## 1.13 Group Action

A group $G$ is said to act on a space $X$ when there is a map $\phi : G \times X \to X$ such that the following conditions hold for all elements $x \in X$.

1. $\phi(e, x) = x$ where $e$ is the identity element of $G$.

2. $\phi(g, \phi(h, x)) = \phi(gh, x)$ for all $g, h \in G$.

In this case, $G$ is called a *transformation group*, $x$ is called a $G$-set, and $\phi$ is called the group action. For a given $x$, the set $\{\phi(g, x) | g \in G\}$, where the group action moves $x$, is called the *orbit* of $x$.

# CHAPTER 2
# MAPS BETWEEN TOPOLOGICAL GROUPS

The group of maps $C(X,Y)$ from a locally compact space $X$ to topological group $Y$ is a topological group if equipped with the compact open topology. The subbasis for this topology consists of sets $[K,U] := \{f \in C(X,Y) | f(K) \subseteq U\}$, where $K$ is a compact subset of $X$ and $U$ is an open subset of $Y$. In addition, if $X$ and $Y$ are Abelian topological groups then we can define the subgroup $\mathrm{Hom}(X,Y)$ of all continuous group homomorphisms inside of $C(X,Y)$. The *dual* or the *character group* of a locally compact Abelian topological group $X$ is the group $X^* := \mathrm{Hom}(X, S^1)$.

## 2.1 Duality for Locally Compact Groups.

**Theorem 2.1.1** *Let $G$ be a locally compact group, let $G^*$ be the dual of $G$ and $G^{**}$ the dual of $G^*$. Then the natural homomorphism $\omega : G \longrightarrow G^{**}$, given by $g \mapsto g^{**}$ where $g^{**}(\xi) := \xi(g)$, for each $\xi \in G^*$, is an isomorphism*

**Proof:** There is a proof in Pontryagin [25]. ∎

## 2.2 Maps between Topological Groups that are Homotopic to Homomorphisms

Given locally compact Abelian groups $G$ and $H$. Let $C_e(G,H)$ be the subset of $C(X,Y)$ that consists of maps that carry the identity element in $G$ to the identity element in $H$. The following theorem was proved by Scheffer [26, Corollary 2 of Theorem 2].

**Theorem 2.2.1** *Let $G$ be a compact connected topological group, and let $H$ be a locally compact Abelian topological group. Then each $f \in C_e(G,H)$ is homotopic to exactly one $h \in \mathrm{Hom}(G,H)$. Furthermore, the homotopy can be chosen to preserve the identity elements.*

**Corollary 2.2.2** *Let $G$ be a compact connected Abelian topological group. Then each homeomorphism $h : (G, e) \longrightarrow (G, e)$ is homotopic to exactly one (group) automorphism.*

**Proof:** By the hypotheses on $G$, there are $\alpha, \beta \in \mathrm{Hom}\,(G, G)$ with $\alpha$ homotopic to $h$ and $\beta$ homotopic to $h^{-1}$. Then $id_G = h \circ h^{-1}$ is homotopic to $\alpha \circ \beta \in \mathrm{Hom}\,(G, G)$. Since there is one element homotopic to $h \circ h^{-1}$ and since $id_G = h \circ h^{-1} \in \mathrm{Hom}\,(G, G)$, $\alpha \circ \beta = id_G$ and that $\beta \circ \alpha = id_G$. We conclude that $\alpha$ is an automorphism which is unique by the theorem above. ∎

## 2.3 The Čech Cohomology of Continua

It is known that every continuum can be expressed as an inverse limit of polyhedra. For the definition of *polyhedra* see Spanier [27, page 113]. Simplicial cohomology theory is defined for polyhedra and is a well developed theory [27]. For definitions of the Čech cohomology functor see Spanier [27, chapter 6]. We shall use the functorial properties of simplicial cohomology to compute the cohomology of inverse limit spaces.

## 2.4 Direct Limits of Groups

A *direct sequence* of Abelian groups and homomorphisms, is a family $(G_i)_{i=1}^{\infty}$ of Abelian groups along with a family of homomorphisms

$$f_{mn} : G_m \longrightarrow G_n,$$

defined for each pair of indices such that $m < n$. We further require that

1. Each $f_{mm} : G_m \longrightarrow G_m$ is the identity;

2. If $l < m < n$, then $f_{mn} \circ f_{lm} = f_{ln}$.

Given a direct sequence of Abelian groups and homomorphisms, we define a group called the *direct limit* of this system as follows. Take the disjoint union of

the groups $G_n$ and identify element $g_n \in G_n$ with $g_m \in G_m$ if, for some integer $u \geq \max\{m,n\}$, we have that

$$f_{mu}(g_m) = f_{nu}(g_n).$$

The direct limit set is the set of equivalence classes. We write $[g_n]$ for the class of the element $g_n \in G_n$ It is made into a group by defining

$$[g_n] + [g_m] = [f_{nu}(g_n) + f_{mu}(g_m)],$$

where $u \geq \max\{m,n\}$.

Suppose one has a sequence of Abelian groups and homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \xrightarrow{f_3} \ldots.$$

It becomes a direct sequence if we define

$$f_{mn} := f_{n-1} \circ f_{n-2} \circ \cdots \circ f_m.$$

## 2.5 The Čech cohomology as a direct limit group

**Theorem 2.5.1** *Let $X$ be a compact triangulable space. Let $D_1 \supset D_2 \supset \ldots$ be a sequence of polyhedra in $X$ whose intersection is $Y$. Let $\check{H}^k(Y;G)$ denote the $k-th$ Čech cohomology group of $Y$ with coefficients in group $G$. Fix $k$ and define*

$$\check{H}^k(Y;G) \approxeq \varinjlim H^k(D_n;G).$$

*Here $\varinjlim H^k(D_n;G)$ denotes the direct limit of the simplicial homology groups $\{H^k(D_n;G)\}_{n=1}^{\infty}$ with bonding maps $i_n^* : H^k(D_n;G) \longrightarrow H^k(D_{n+1};G)$ induced by the inclusion $i_n : D_{n+1} \hookrightarrow D_n$ at cohomology level.*

**Proof:** There is a proof of this theorem in Spanier [27, page 359]. ∎

# CHAPTER 3
# FIXED POINTS OF KNASTER CONTINUA

We construct a continuum on which every homeomorphism has either uncountably many fixed points or uncountably many points of period two. We also show that for generalized one-dimensional Knaster continua, with the exception of those Knaster continua in which the prime 2 occurs infinitely often, there is always a homeomorphism with a single fixed point. This contradicts the claim in the paper of Aarts and Fokkink [2, Theorem 15]. In addition, we give necessary and sufficient conditions for homeomorphisms to have at least two fixed points on generalized Knaster continua. Finally, for a given generalized Knaster continua, we give a lower bound on the number of fixed points of an arbitrary homeomorphism.

## 3.1 Generalized Solenoids

Let $\alpha = (p_1, p_2, p_3, \dots)$ be a sequence of prime numbers. Define the $\alpha$-adic solenoid $\mathcal{S}_\alpha$ as the inverse limit space of mappings $z \longmapsto z^{p_k}$ for $k = 1, 2, \dots$, on the complex circle $S^1 := \{z \in \mathbb{C} : |z| = 1\}$. Write

$$\mathcal{S}_\alpha = \varprojlim(S^1; z \longmapsto z^{p_k}) := \left\{(z_k)_{k=1}^{\infty} : z_k = z_{k+1}^{p_k}, \text{for each } k \geq 1\right\}.$$

This $\mathcal{S}_\alpha$ is a closed connected subgroup of the product $\prod_{k=1}^{\infty} S^1$ and is therefore a compact Abelian topological group. We write $z \cdot \tilde{z}$ for the product of its two elements $z$ and $\tilde{z}$ and use $e$ for the identity element $(1, 1, 1, \dots)$.

The $\alpha$-adic solenoid can be visualized as an intersection of a nested sequence of progressively thiner solid tori that are each wrapped into the previous torus a number

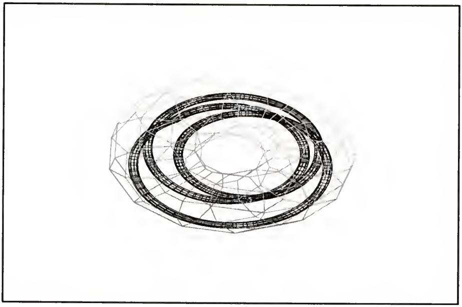

Figure 3.1: A torus wrapped 3 times and put inside itself marks the beginning of the construction of the (3,2,2, ...) solenoid.

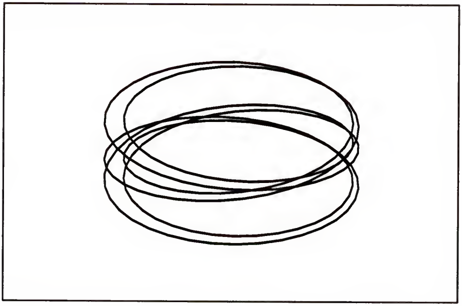

Figure 3.2: Stage two of approximation of the (3,2,2, ...) solenoid.

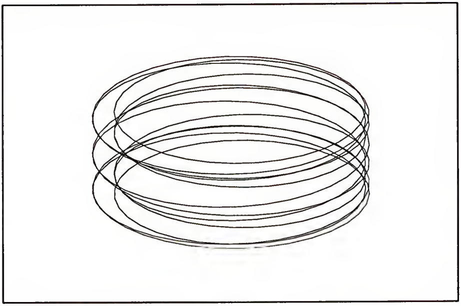

Figure 3.3: Stage three of the construction of the (3,2,2, ...) solenoid.

of times (see figure 3.1) determined by the sequence $\alpha$. The limiting intersection corresponds to a specific embedding of $\mathcal{S}_\alpha$ in 3-space.

## 3.2 Composants of the Solenoid

Let $\rho : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ denote the involution $z \longmapsto z^{-1}$. The exponential covering map $\exp(t) = e^{2\pi i t}$ from the real numbers $\mathbb{R}$ to $S^1$ makes the diagram

$$\begin{array}{ccccccc} R & \xleftarrow{\times p_1} & R & \xleftarrow{\times p_2} & R & \xleftarrow{\times p_3} & \dots \\ {\scriptstyle exp}\downarrow & & {\scriptstyle exp}\downarrow & & {\scriptstyle exp}\downarrow & & \\ S^1 & \xleftarrow{z^{p_1}} & S^1 & \xleftarrow{z^{p_2}} & S^1 & \xleftarrow{z^{p_3}} & \dots \end{array}$$

commute and therefore induces a map $s : \mathbb{R} \longrightarrow \mathcal{S}_\alpha$ (in the diagram $R$ is used instead of $\mathbb{R}$). This map is a homomorphism with dense image, i.e., $\overline{s(\mathbb{R})} = \mathcal{S}_\alpha$. Here the closure operation is taken in $\mathcal{S}_\alpha$. The map $s$ is also one to one onto its image. Denote $s(\mathbb{R})$ by $\Gamma$. The space $\Gamma$ is a subgroup of $\mathcal{S}_\alpha$ and is the composant of the identity element $e$. All other composants are cosets, in $\mathcal{S}_\alpha$, of the subgroup $\Gamma$. In particular the composant of a point $p \in \mathcal{S}_\alpha$ is $p \cdot \Gamma$. Moreover

$$\Gamma = \left\{ \left( \exp\left( \frac{2\pi i t}{p_1 \dots p_k} \right) \right)_{k=1}^{\infty} : t \in \mathbb{R} \right\}.$$

Define $T : \mathbb{R}\times\mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ by $T(t,z) := s(t)\cdot z$. Then $T$ is a *flow* i.e., $T$ is continuous and satisfies the following property: $T(s+t,z) = T(s,T(t,z))$ for every $s,t\in\mathbb{R}$. Furthermore, $T(0,z) = z$ for all $z\in\mathcal{S}_\alpha$. Let

$$\Lambda_\alpha := \{z = (z_k)_{k=0}^{\infty} \in \mathcal{S}_\alpha : z_0 = 1\}.$$

The solenoid $\mathcal{S}_\alpha$ can be realized as the orbit space of the group action of the integers $\mathbb{Z}$ on $\Lambda_\alpha\times\mathbb{R}$ via the map $\phi : \mathbb{Z}\times(\Lambda_\alpha\times\mathbb{R})\longrightarrow\Lambda_\alpha\times\mathbb{R}$ given by $(k,(\omega,t))\longmapsto (T(\omega,k), t-k)$; see Kwapisz [17]. Let $p$ denote the natural projection

$$p : \Lambda_\alpha\times\mathbb{R}\longrightarrow(\Lambda_\alpha\times\mathbb{R})/I \quad\cong\quad \mathcal{S}_\alpha,$$

where $I$ is the subgroup $I = \left\{\left(\gamma^k, t-k\right) : k\in\mathbb{Z}\right\}$. Topologically, $\Lambda_\alpha$ is a Cantor set homeomorphic to $\prod_{k=1}^{\infty}\{0,\dots p_k - 1\}$ where

$$(d_k)_{k=1}^{\infty}\longmapsto\left(\exp\left(2\pi i\frac{d_1+d_2p_1+d_3p_1p_2+\cdots+d_kp_1\dots p_{k-1}}{p_0p_1\dots p_k}\right)\right)_{k=1}^{\infty}.$$

We shall use the metric $d_{\Lambda_\alpha}$ on $\Lambda_\alpha$ given by

$$d_{\Lambda_\alpha}\left((d_k)_{k=1}^{\infty},\left(\tilde{d}_k\right)_{k=1}^{\infty}\right) = \exp\left(-\min\left\{k\geq 1 : d_k\neq\tilde{d}_k;\infty\right\}\right).$$

The metric $d_{\Lambda_\alpha}$ is induced by the norm $\|(d_k)_{k=1}^{\infty}\| := \exp(-\min\{k : d_k\neq 0\})$ and therefore is an invariant metric (i.e., the translations of $\Lambda_\alpha$ are isometries). This induces metrics on $\Lambda_\alpha\times\mathbb{R}$ and $\mathcal{S}_\alpha$ given by

$$d_{\Lambda_\alpha\times\mathbb{R}}((\varpi,x),\ (\tilde{\varpi},\tilde{x})) = \max\{d_{\Lambda_\alpha}(\varpi,x),\ |x-\tilde{x}|\}$$

and

$$d_{\mathcal{S}_\alpha}(z,\tilde{z}) = \min\{d_{\Lambda_\alpha\times\mathbb{R}}((\varpi,x),(\tilde{\varpi},\tilde{x})) : p(\varpi,x) = z,\quad p(\tilde{\varpi},\tilde{x}) = \tilde{z}\}.$$

The covering projection becomes a local isometry.

### 3.3 Knaster Continua

Define the $\alpha$-adic Knaster continuum as the quotient $\mathcal{S}_\alpha$ by the relation $\backsim$ where $\bar{z} \backsim \bar{z}'$ iff $\bar{z}' = \bar{z}^{-1}$. Denote the quotient map by $\pi : \mathcal{S}_\alpha \longrightarrow \mathcal{K}_\alpha$. We give $\mathcal{K}_\alpha$ the metric

$$d_{\mathcal{K}_\alpha}(u,v) := \min\{d_{\mathcal{S}_\alpha}(w,z) : \pi(w) = u,\ \pi(z) = v, \quad \text{for each pair } w, z \in \mathcal{S}_\alpha\}.$$

The projection $\pi$ can also be realized by taking the limit of the vertical maps in this diagram

$$\begin{array}{ccccccc}
S^1 & \xleftarrow{z^{p_1}} & S^1 & \xleftarrow{z^{p_2}} & S^1 & \xleftarrow{z^{p_3}} & \cdots \\
{\scriptstyle Re(z)}\downarrow & & {\scriptstyle Re(z)}\downarrow & & {\scriptstyle Re(z)}\downarrow & & \\
I & \xleftarrow{f_1} & I & \xleftarrow{f_2} & I & \xleftarrow{f_3} & \cdots
\end{array}$$

where $I = [-1,1]$ and $\{f_n\}_{n=1}^{\infty}$ are the Chebychev polynomials, $f_n = T(p_n, x)$ of degree $p_n$.

### 3.4 Chebychev polynomials

For a given complex number $z$, the $n^{th}$ degree Chebychev polynomial assigns the real part $z$ to real part of $z^n$. So that if we write $z = x + iy$, the

$$T(n,x) = Re(z^n) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} x^{n-2k} \left(x^2 - 1\right)^k.$$

Each $f_n$ has exactly $p_n$ monotonic laps and each $f_n$ is surjective on the interval $[-1,1]$. Note that the Chebychev polynomials commute (under composition) with each other on the interval $[-1,1]$. Also the even degree Chebychev polynomials are even functions and the odd degree Chebychev polynomials are odd functions.

### 3.5 End Points in Knaster Continua

The projection $\pi : \mathcal{S}_\alpha \longrightarrow \mathcal{K}_\alpha$ is 2 to 1 except at the elements of $\mathcal{S}_\alpha$ that are idempotents. There are two possible cases. The first case is when $\alpha$ contains infinitely many 2's. In this case there are no elements of degree two.. In the second

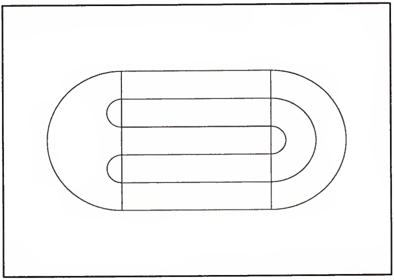

Figure 3.4: Two embedded disks whose intersection forms the first approximation to the dyadic Knaster continuum. This diffeomorphism is also known as the Horseshoe map.

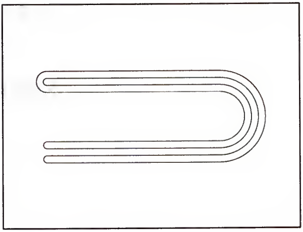

Figure 3.5: The second approximation to the dyadic Knaster continuum

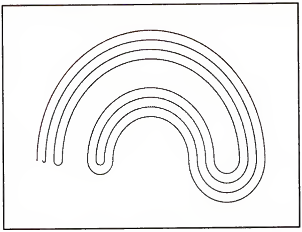

Figure 3.6: The dyadic Knaster continuum: Suppose that $C$ denotes the Cantor middle-thirds set sitting on the unit interval $[0,1] \times \{0\}$ in the plane. Connect the points with semicircles as follows: (1) For each pair $p, q$ of points of $C$ that are equidistant from $(1/2, 0)$, connect $p$ and $q$ with a semicircle sitting above the $x$-axis. (2) For each pair $p, q$ of points of $C$ equidistant from $(5/6, 0)$, connect $p$ and $q$ with a semicircle extending below the $x$-axis. (3) For each pair $p, q$ of points equidistant from the midpoint $(5/18, 0)$ of $(2/9, 0)$ and $(1/3, 0)$, connect $p$ and $q$ with a semicircle that extends below the $x$-axis. Continue this processs. The nth step consists of connecting each pair $p, q$ of points equidistant from the midpoint $(5/3^n, 0)$ of the points $(2/3^n, 0)$ and $(1/3^n, 0)$ with a semicircle that extends below the $x$-axis.

case there are only a finite number of 2's in $\alpha$. This leads to two idempotents $e$ and $e^* := (1, \ldots 1, -1, -1, \ldots)$ where the number $r$ of the leading 1's in the sequence is such that $r = \max\{k : p_k = 2\}$. In case all the primes are odd, there are no leading 1's and $e^* = (-1, -1, \ldots)$.

These two cases lead to the following:

**Lemma 3.5.1** *The Knaster continuum $\mathcal{K}_\alpha$ has either one or two endpoints.*

A given Knaster continuum $\mathcal{K}_\alpha$ can be embedded in the plane as an intersection of a nested sequence of disks each traversing the previous one in a snake-like fashion a number of times indicated by the corresponding term in $\alpha$ (figure 3.4-3.7).

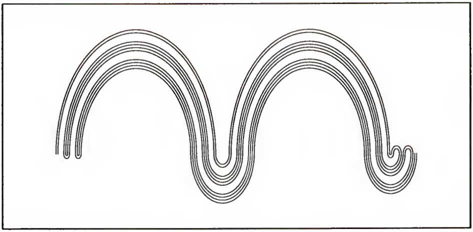

Figure 3.7: The 5-adic Knaster Continuum.

## 3.6 Fixed Points of Homeomorphisms of Knaster Continua

In response to a question of Mahavier [9, page 384, problem 120], Aarts and Fokkink [2] considered the Knaster bucket handle $\mathcal{K}_{(2,2,\ldots)}$ (write $\mathcal{K}_2$). They determined that every homeomorphism on this continuum has at least two fixed points. They claimed that, in general, for a given prime $p$ and any homeomorphism $g : \mathcal{K}_p \longrightarrow \mathcal{K}_p$, writing $\mathcal{K}_p$ for $\mathcal{K}_{(p,p,\ldots)}$, that the number elements in the set $\{x | g^n(x) = x\}$ is at least $p \cdot n$. We exhibit a homeomorphism of $\mathcal{K}_3$ that has a single fixed point with all the other points being of period two. In particular this gives a counterexample to the authors' claim. In addition, we give minimal estimates for the cardinalities of periodic orbits of the most general Knaster continua

For the promised example consider the homeomorphism $g : \mathcal{K}_3 \longrightarrow \mathcal{K}_3$ defined by $g(x_1, x_2, x_3, \ldots) := (-x_1, -x_2, -x_3, \ldots)$. To see that this is a homeomorphism, observe that the bonding map that yields $\mathcal{K}_3$ is $f := x \longmapsto 4x^3 - 3x$. This map commutes with $h : I \longrightarrow I$, $x \longmapsto -x$. So we have the following commuting diagram

$$\begin{tikzcd}
I \arrow[d, "h"'] & I \arrow[l, "f"'] \arrow[d, "h"'] & I \arrow[l, "f"'] \arrow[d, "h"'] & \cdots \arrow[l, "I"'] \\
I & I \arrow[l, "f"'] & I \arrow[l, "f"'] & \cdots \arrow[l, "f"']
\end{tikzcd}$$

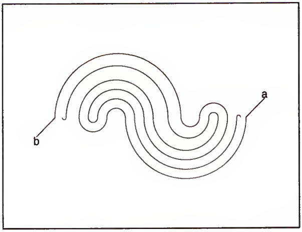


Figure 3.8: Triadic Knaster continuum. The points a and b are the end points.

The map g is induced on the limit by the homeomorphism $h$. Clearly $g$ has the sole fixed point $(0, 0, \dots)$, and all the other points are of period two. In particular $g$ is a periodic homeomorphism of period 2.

In general, this homeomorphism exists on any odd p-adic Knaster continuum. In a situation when the Knaster continuum has two end points, it is conceivable that there exists a "rotational" homeomorphism with a single fixed point at the center of rotation. This turns out to be the case. However, as we show later, the condition of having a single end point is not necessary for a homeomorphism to have at least two fixed points. We exhibit a Knaster continuum with one end point and a homeomorphism on it that has a single fixed point.

Let $\alpha$ and $\beta$ be an infinite sequence of primes. We say $\alpha \backsim \beta$ if and only if a finite number of primes can be deleted from both $\alpha$ and $\beta$ so that the primes remaining in both sequence occur the same number of times.

**Theorem 3.6.1** *Let $\alpha$ and $\beta$ be infinite sequences of primes, allowing repetition. Then $\alpha \backsim \beta$ if and only iff the solenoids $\mathcal{S}_\alpha$ and $\mathcal{S}_\beta$ are homeomorphic.*

**Proof:** There is a proof in [2]. ∎

Let $\mathbb{P}$ be the subset of the positive integers $\mathbb{N}$, that consists of all prime numbers. Let $\alpha = (p_1, p_2, p_3, \ldots)$ be a sequence of primes. Define the multiplicity function $m_\alpha : \mathbb{P} \longrightarrow \mathbb{N}$ by $m_\alpha(p) =$ the number of occurrences of the prime $p$ in the sequence $\alpha$. A prime $p$ is recurrent in $\alpha$ if $m_\alpha(p) = \infty$.

**Lemma 3.6.2** *Let $f_p$ be the p-th degree Chebychev polynomial, $p \in \mathbb{P}$ and let $FPS(f_p)$ be the set of fixed points of $f_p$ in $[-1,1]$. Then $\left\{-\frac{1}{2}, 1\right\} \subseteq FPS(f_p)$ iff $p \neq 3$.*

**Proof:** Clearly $f_3(-\frac{1}{2}) \neq -\frac{1}{2}$. For the other implication, let $z := e^{i\frac{2\pi}{3}}$ and consider a prime $p \neq 3$. Then either $p \equiv 1 (\mathrm{mod}\, 3)$ or $p \equiv 2 (\mathrm{mod}\, 3)$. Suppose $p \equiv 1 (\mathrm{mod}\, 3)$, then for some integer $k > 1$,

$$z^p = e^{i\frac{2\pi p}{3}} = e^{i\frac{2\pi(3k+1)}{3}} = e^{i2\pi k} e^{i\frac{2\pi}{3}} = e^{i\frac{2\pi}{3}} = z.$$

Since $\mathrm{Re}(z) = \frac{-1}{2}$ it follows that $f_p(-\frac{1}{2}) = -\frac{1}{2}$. Suppose on the other hand that $p \equiv 2 (\mathrm{mod}\, 3)$ then for some $k \geq 0, z^p = e^{i\frac{2\pi p}{3}} = e^{i\frac{2\pi(3k+2)}{3}} = e^{i2\pi k} e^{i\frac{4\pi}{3}} = e^{i\frac{4\pi}{3}}$ and $\mathrm{Re}\left(e^{i\frac{4\pi}{3}}\right) = -\frac{1}{2}$ and therefore $f_p(-\frac{1}{2}) = -\frac{1}{2}$. Since $1^p = 1$ for every $p$, the result follows. ■

The following theorem shows that the requirement for a Knaster continuum to have exactly one endpoint is not sufficient for every homeomorphism on that Knaster continuum to have at least two fixed points.

**Theorem 3.6.3** *Let $\alpha = (p_1, p_2, p_3, \ldots)$ be a sequence of primes, such that $m_\alpha(3) = m_\alpha(2) = \infty$. Then there exists a homeomorphism $h : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ with exactly one fixed point.*

**Proof:** Observe that the following diagram commutes

$$\begin{array}{ccccccc}
I & \xleftarrow{f_{p_1}} & I & \xleftarrow{f_{p_2}} & I & \xleftarrow{f_{p_3}} & \cdots \\
{\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & {\scriptstyle f_2}\big\downarrow & & \\
I & \xleftarrow{f_{p_1}} & I & \xleftarrow{f_{p_2}} & I & \xleftarrow{f_{p_3}} & \cdots
\end{array}$$

and induces a homeomorphism of $\mathcal{K}_\alpha$ to $\mathcal{K}_\alpha$,

$$\hat{f}_2 := (x_1, x_2, x_3, \dots) \mapsto (f_2(x_1), f_2(x_2), f_2(x_3), \dots)$$

where $f_2(x) = 2x^2 - 1$. Clearly, $(1, 1, 1, \dots)$ is a fixed point of $\hat{f}_2$. We shall show this is the only fixed point. Suppose $(f_2(x_1), f_2(x_2), f_2(x_3), \dots) = (x_1, x_2, x_3, \dots)$, then $x_i \in \left\{-\frac{1}{2}, 1\right\} = FPS(f_2)$ for all $i$ . Chose $k$ such that $p_k = 3$ in $\alpha$. Then $x_k = 1$ since $f_3^{-1}\{-\frac{1}{2}\} \cap \left\{-\frac{1}{2}, 1\right\} = \emptyset$. It follows that $x_i = 1$ for all $i \leq k$. Since there are infinitely many such $k's$, $x_i = 1$ for all $i$. So $\hat{f}_2$ has a single fixed.Hence $h := \hat{f}_2$ is the required homeomorphism. ∎

**Theorem 3.6.4** *Let $\alpha = (p_1, p_2, p_3, \dots)$ be a sequence of primes such that $m_\alpha(3) \neq \infty and m_\alpha(2) = \infty$. Then every homeomorphism $g : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ has at least two fixed points.*

The proof of the above theorem and other results will easily follow from some background results that we shall now give. We observe that many of these results are now part of folklore. However, some of the results do not seem to have complete proofs in the literature. For that reason we shall include complete proofs where necessary. The space $\mathcal{S}_\alpha$ is a compact connected abelian topological group. We now study $\mathbf{Aut}(\mathcal{S}_\alpha)$, the group of all topological group automorphisms of $\mathcal{S}_\alpha$ with the compact open topology. By Pontryagin duality, $\mathbf{Aut}(\mathcal{S}_\alpha)$ is isomorphic to the group all automorphisms of the dual, $\mathcal{S}_\alpha^*$, of $\mathcal{S}_\alpha$. In turn due to a theorem of Steenrod [15], $\mathcal{S}_\alpha^*$ is isomorphic to $\check{H}^1(\mathcal{S}_\alpha)$. Here $\check{H}^1(\mathcal{S}_\alpha)$ is the one-dimensional Čech cohomology group with integer coefficients of $\mathcal{S}_\alpha$. In addition $\mathbf{Aut}(\mathcal{S}_\alpha)$ is isomorphic to $\mathbf{Aut}(\mathcal{S}_\alpha^*)$ since if $\eta : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ is an automorphism of $\mathcal{S}_\alpha$ then there is an automorphism $\eta^* : \mathcal{S}_\alpha^* \longrightarrow \mathcal{S}_\alpha^*$ that is the dual of the automorphism $\eta$. In particular for a character $\xi \in \mathcal{S}_\alpha^*$, $\eta^*(\xi)(x) = \xi(\eta(x))$, for $x \in \mathcal{S}_\alpha$. So we therefore have $\mathbf{Aut}(\mathcal{S}_\alpha) \cong \mathbf{Aut}(\mathcal{S}_\alpha^*) \cong \mathbf{Aut}\left(\check{H}^1(\mathcal{S}_\alpha)\right)$. We shall therefore first compute $\check{H}^1(\mathcal{S}_\alpha)$. We shall use one of the special features of the Čech groups - the continuity property

[11, capter X] - that allows the cohomology groups of $\mathcal{S}_\alpha$ to be computed by taking direct limits of the Čech cohomology groups of $S^1$, that happen to coincide with the simplicial cohomology groups of $S^1$

**Theorem 3.6.5** *Let $\alpha = (p_1, p_2, p_3, \dots)$ be a sequence of primes and $\mathcal{S}_\alpha$ be the $\alpha$-adic solenoid. Then $\check{H}^1(\mathcal{S}_\alpha)$ is isomorphic to the group of $\alpha-$adic rationals $\mathbb{Q}_\alpha$ i.e. rationals of the form $\frac{m}{p_{k_1}^{n_1} p_{k_2}^{n_2} \cdots p_{k_r}^{n_r}}$, where $p_{k_1}, p_{k_2}, \dots, p_{k_r}$ are primes from $\alpha$ and $k_1, k_2, \dots k_r, n_1, n_2, \dots n_r$ are integers and $n_i \le m_\alpha(p_{k_i})$ for $i = 1, \dots, r$.*

**Proof:** By applying $H^1(\cdot, \mathbb{Z})$ (the simplicial cohomology contravariant functor with integer coefficients) to the following diagram

$$S^1 \xleftarrow{z^{p_1}} S^1 \xleftarrow{z^{p_2}} S^1 \xleftarrow{z^{p_3}} \dots$$

we get the diagram

$$Z \xrightarrow{\times p_1} Z \xrightarrow{\times p_2} Z \xrightarrow{\times p_3} \dots$$

where $\times p_k$ is the multiplication by $p_k$ induced at cohomology level by the map $z \longmapsto z^{p_k}$. It follows that $\check{H}^1(\mathcal{S}_\alpha) = \varinjlim(\mathbb{Z}, p_k)$ which is readily seen to be $\mathbb{Q}_\alpha$. For instance define

$$\phi_n : \mathbb{Z} \longrightarrow \mathbb{Q}_\alpha$$

by $\phi_n(m) = \frac{m}{p_1 p_2 \cdots p_{n-1}}$. Then denoting by $f_n : m \longmapsto m \times p_n$ we see that $\phi_n \circ f_{n-1} = \phi_{n-1}$. The map $\Phi := \{\phi_n\}_{n=1}^{\infty}$ induces a direct limit homomorphism $\underrightarrow{\Phi}$ : $\varinjlim(\mathbb{Z}, p_k) \longrightarrow \mathbb{Q}_\alpha$ where $\underrightarrow{\Phi}([g_n]) \mapsto [\phi_n(g_n)]$. It is easy to check that this map is both injective and surjective and therefore an isomorphism. ■

Keesling [15] proved the followinf result.

**Lemma 3.6.6** *Let $\mathbb{Z}$ be the group of integers and $\mathbb{Z}_2$ be the group of integers modulo 2. Let $n$ be the number of prime numbers $p$ such that $m_\alpha(p) = \infty$. Then the group **Aut**$(\mathcal{S}_\alpha)$ of automorphisms of $\mathcal{S}_\alpha$ is isomorphic to (a) $\mathbb{Z}_2$ if $n = 0$, (b) $\mathbb{Z}_2 \times \mathbb{Z}^n$ if $n$ is a positive integer, and (c) $\mathbb{Z}_2 \times \bigoplus_{i=1}^{\infty} \mathbb{Z}$ if $n$ is infinite.*

## 3.7 Standard Homeomorphisms of the Solenoid

We shall exhibit standard homeomorphisms that will work as the generators for $\mathbf{Aut}(\mathcal{S}_\alpha)$. Observe that if $p \in \alpha$ and $m_\alpha(p) = \infty$ then the map $\eta_p : \mathbb{Q}_\alpha \longrightarrow \mathbb{Q}_\alpha$ given by $\eta_p(q) = qp$ is an automorphism. Note that if $m_\alpha(p) \neq \infty$, then the map is not invertible. So for each prime $p \in \alpha$ with $m_\alpha(p) = \infty$, $\eta_p$ is a generator of a copy of $\mathbb{Z}$ in $\mathbf{Aut}(\mathbb{Q}_\alpha)$. The $\mathbb{Z}_2$ component is generated by the involution $q \mapsto q^{-1}$. Analogously, for the corresponding solenoid, $\mathcal{S}_\alpha$, the automorphisms induced by the maps $z \mapsto z^p$ on $\mathcal{S}_\alpha$ is the corresponding generator of a copy $\mathbb{Z}$ in $\mathbf{Aut}(\mathcal{S}_\alpha)$ and the involution $q \mapsto q^{-1}$generates the $\mathbb{Z}_2$ component. If $a$ is a product of prime numbers from $\alpha$ each of which appears infinitely often in $\alpha$, we shall denote by $g_a$ the automorphism on $\mathcal{S}_\alpha$ induced by the map $z \mapsto z^a$. Write $g_{\frac{a}{b}}$ for the composition $g_a \circ g_b^{-1}$, where $b$ is a product of primes in $\alpha$. Recall that $\rho : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ is the involution on the solenoid. The standard homeomorphisms of $\mathcal{S}_\alpha$ will be either of the form $g_{\frac{a}{b}}$ or of the form $\rho \circ g_{\frac{a}{b}}$.

We know from Theorem 2.2.1 that every homeomorphism of $\mathcal{S}_\alpha$ that fixes the identity element is homotopic to an automorphism. In particular, every homeomorphism of $\mathcal{S}_\alpha$ that leaves the identity element fixed, permutes the composants in exactly the same way as one of the standard homeomorphisms.

## 3.8 Lifting Maps from Knaster Continua to the Solenoids

**Proposition 3.8.1** *For any homeomorphism $f : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ there exists homeomorphisms $\hat{f}_1, \hat{f}_2 : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ that are lifts of $f$ i.e. $\pi \circ \hat{f}_i = f \circ \pi$ for $i = 1, 2$.*

**Proof:** There is a proof in Kwapisz [17]. ■

## 3.9 Lifting Isotopies.

Let $g, h : (X, p) \longrightarrow (X, p)$ be homeomorphisms of the pointed space $(X, p)$. Then $h$ and $g$ are said to be isotopic to each other if there exists a map $H : X \times I \longrightarrow$

$X$ such that $H(\cdot,t) : X \longrightarrow X$ is a homeomorphism for all $t \in I$, $H(\cdot,0) = h$, $H(\cdot,1) = g$ and $H(p,t) = p$ for all $t \in I$.

**Theorem 3.9.1** *Let $H : \mathcal{K}_\alpha \times I \longrightarrow \mathcal{K}_\alpha$ be an isotopy that fixes the endpoints of $\mathcal{K}_\alpha$ at every level $t \in I$ i.e given an endpoint $x \in \mathcal{K}_\alpha$, $H(x,t) = x$ for all $t \in I$. Then there exists an isotopy $\tilde{H} : \mathcal{S}_\alpha \times I \longrightarrow \mathcal{S}_\alpha$ such that $H \circ \pi = \pi \circ \tilde{H}$.*

**Proof:** Assume that $H$ fixes the endpoints of $\mathcal{K}_\alpha$ and consider the restriction of $H$ to $\mathcal{K} = \mathcal{K}_a \setminus \{\text{endpoints}\}$. $H$ may now be considered an isotopy on $\mathcal{K}$. Let $\mathcal{S}$ be $\pi^{-1}(\mathcal{K})$. By [?, Propositoin 9.3, JK]here is a homeomorphism $\tilde{h}_0 : \mathcal{S} \longrightarrow \mathcal{S}$ that is the lift of $h_0 = H(\cdot,0)$. Given $(x,t) \in \mathcal{S}_\alpha \times I$, there is a path $A$ that connects $(x,t)$ to $(x,0)$, namely the path $x \times [0,t]$. $H((p \times id_I)(A))$ is a path in $K$. By the unique path lifting property of $p$ there is a unique path $\omega$ that begins at $h_0(x) = H(x,0)$ that is the lift of $H((p \times id_I)(A))$. Define $\tilde{H}(x,t)$ to be the end point of that path.

As before, we can check that every level of $\tilde{H}$ is a homeomorphism and there are two possible such $\tilde{H}$ 's. ∎

In the situation when there are two endpoints $\bar{e}$, $\bar{e}^*$ and $H(\bar{e},t) = \bar{e}^*$, then $g \circ H(\bar{e},t) = \bar{e}$ where g is as defined before. In this case $g \circ H : \mathcal{K}_\alpha \times I \longrightarrow \mathcal{K}_\alpha$ is an isotopy that keeps the endpoints fixed.

**Lemma 3.9.2** *Let $h : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ be a homeomorphism. There are exactly two homeomorphisms $\tilde{h}_1, \tilde{h}_2 : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ of the $\alpha$-adic solenoid that are lifts of $h$.*

**Proof:** By Theorem 3.9.1, $h$ can be lifted to a homeomorphisms $\tilde{h}, \rho \circ \tilde{h} : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$. We shall show that these are the only lifts. The proof follows the aguement in Aarts and Fokkink [2]. Suppose first that $h$ is the identity on $\mathcal{K}_\alpha$ and let $\tilde{h} : \mathcal{S}_\alpha \longrightarrow \mathcal{S}_\alpha$ be its lift. Then

$$\mathcal{S}_\alpha = \left\{x | \tilde{h}(x) = x\right\} \cup \left\{x | \tilde{h}(x) = \rho(x)\right\},$$

which is a union of two closed sets with intersection $\{e\}$. By the connectedness of $\mathcal{S}_\alpha \setminus \{e\}$ one of these sets must be empty. It follows therefore that either $\tilde{h} = id_{\mathcal{S}_\alpha}$ or $\tilde{h} = \rho$. So the statement is true in this case. Now for any other $h$, let $\tilde{h}_1, \tilde{h}_2$ be lifts of $h$. If $\bar{h}$ is a lift of the inverse of $h$, then both $\tilde{h}_1 \circ \bar{h}$ and $\tilde{h}_2 \circ \bar{h}$ are lifts of the identity and the result follows. ∎

As in Aarts and Fokkink [2], a composant of a point $p \in \mathcal{S}_\alpha$ is $\Gamma_p = p \cdot \Gamma$ and if $\mathcal{C}$ is a composant of a point in $\mathcal{K}_\alpha$ that does not contain an end point, then $\mathcal{C}$ lifts to two composants $\tilde{\Gamma}$ and $\rho\left(\tilde{\Gamma}\right)$ in $\mathcal{S}_\alpha$. The proofs of the following lemmas are adopted from lemma 12 and lemma 13 in Aarts and Fokkink [2].

**Lemma 3.9.3** *Let $p \in \mathcal{S}_\alpha$, where $\alpha$ is a sequence of primes. Suppose that $g : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$ is a homeomorphism such that $g(\Gamma_p) \subset \Gamma_p$. Suppose $g(x) \neq x$ for all $x \in \Gamma_p$. Then $(g - id_{\mathcal{S}_\alpha})(\Gamma_p)$ is contained in a compact subset of $\Gamma$.*

**Proof:** There is an automorphism $\gamma : \mathcal{S}_\alpha \to \mathcal{S}_\alpha$ that is isotopic to $g$. For each $x \in \mathcal{S}_\alpha$, $g(x)$ and $\gamma(x)$ belong to the same composant. It follows that $g(x) - \gamma(x) \in \Gamma$ for all $x$. Define $\beta := g - \gamma$ and $\alpha := \gamma - id_{\mathcal{S}_\alpha}$. As the cosets of $\Gamma$ coincide with the composants, $\beta(\mathcal{S}_\alpha)$ is a compact subset of $\Gamma$. Note that $\alpha$ is a continuous homomorphism. There are two cases: $\alpha(\Gamma) = \Gamma$ or $\alpha(\Gamma) = \{0_{\mathcal{S}_\alpha}\}$. In the latter case $\alpha(\mathcal{S}_\alpha) = 0_{\mathcal{S}_\alpha}$, whence $\gamma$ equals $id_{\mathcal{S}_\alpha}$and $\beta = g - id_{\mathcal{S}_\alpha}$. In particular, $g - id_{\mathcal{S}_\alpha}(\Gamma_p) = \beta(\Gamma_p) \subset \beta(\mathcal{S}_\alpha)$, which is a compact subset of $\Gamma$, and the proof is complete. So we may assume that $\alpha(\Gamma) = \Gamma$. Under this assumption by an intermediate value argument we shall show that there exists a $z \in \Gamma$ such that $g(z + p) = z + p$. This is impossible, as $g(x) \neq x$ for all $x \in \Gamma_p$.

Since $g(\Gamma_p) \subset \Gamma_p$, we have $g(p + y) - (p + y) \in \Gamma$ for all $y \in \Gamma$. It follows that for all $y \in \Gamma$,

$$\begin{aligned} g(p+y) - (p+y) &= \gamma(p+y) - (p+y) + \beta(p+y) \\ &= \alpha(p) + \alpha(y) + \beta(p+y). \end{aligned} \tag{3.1}$$

The intermediate value argument follows. As $\beta$ is continuous, $\beta(\mathcal{S}_\alpha)$ is a compact and connected subset of $\Gamma$. Note that $\alpha$ and $\beta(p+\cdot)$ send $\Gamma$ to $\Gamma$. So there are continuous lifts $\tilde{\alpha} : \mathbb{R} \to \mathbb{R}$ and $\tilde{\beta}' : \mathbb{R} \to \mathbb{R}$ of $\alpha$ and $\beta(p+\cdot)$ respectively. As $\tilde{\alpha}(\mathbb{R}) = \mathbb{R}$ and $\tilde{\beta}'(\mathbb{R})$ is compact, $\tilde{\alpha} + \tilde{\beta}'$ maps $\mathbb{R}$ onto itself. It follows that for some $x$ in $\mathbb{R}$ we have, writing $s(x) = z$,

$$s \circ (\tilde{\alpha} + \tilde{\beta}')(x) = \alpha(z) + \beta(p+z) = -\alpha(p).$$

It follows that $(g - id_{\mathcal{S}_\alpha})(p+z) = 0_{\mathcal{S}_\alpha}$. ∎

**Lemma 3.9.4** *If a homeomorphism on $\mathcal{K}_\alpha$ has an invariant composant, it has a fixed point in that composant.*

**Proof:** Suppose that $g : \mathcal{K}_\alpha \to \mathcal{K}_\alpha$ has an invariant composant $\Gamma_p$. We may assume that $\Gamma_p$ is not a composant of one of the end points of $\mathcal{K}_\alpha$. The preimage of $\Gamma_p$ in $\mathcal{S}_\alpha$ under $\pi$ is the union of two composants $\Gamma_1, \Gamma_2$ of $\mathcal{S}_\alpha$, which are interchanged by the involution$\rho$. Let $\tilde{g}$ be the lift of $g$ which has both $\Gamma_1$ and $\Gamma_2$ as an invariant set. We show that $\tilde{g}$ has a fixed point of $\Gamma_1 \cup \Gamma_2$. The proof is by contradiction. If there is no fixed point, then by Lemma 3.9.3 the map $\tilde{h} = \tilde{g} - id_{\mathcal{K}_\alpha}$ sends $\Gamma_1$ into a compact subset of $\Gamma$. The same is true for $\Gamma_2$. Observe that $\tilde{h} \circ \rho = \rho \circ \tilde{h}$, so $A := \tilde{h}(\Gamma_1 \cup \Gamma_2)$ is invariant under $\rho$. Note that $A$ is contained in a compact subset $K$ of $\Gamma$. Furthermore, $A$ is connected, since $\Gamma_1 \cup \Gamma_2$ is connected. So we may assume that $K$ is connected as well. So there is a compact interval $J$ in $\mathbb{R}$ such that $s(J) = K$. The restriction of $s^{-1}$ to $K$ is a homeomorphism. It follows that $s^{-1}(A)$ is a connected subset of $\mathbb{R}$ such that $s^{-1}(A) = -s^{-1}(A)$. Consequently $0 \in s^{-1}(A)$. Hence $0_{\mathcal{S}_\alpha} \in A$, so $\tilde{h}$ has a fixed point in $\Gamma_1 \cup \Gamma_2$. ∎

## 3.10 Standard Homeomorphisms of Knaster Continua

Let $\alpha = (p_1, p_2, p_3, \ldots)$ be a sequence of primes and let $f_{p_i}$ be the $p_i-$degree Chebychev polynomial, then $\hat{f}_{p_i} : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ defined by $\hat{f}_{p_i}(x_1, x_2, x_3, \ldots) =$

$(f_{p_i}(x_1), f_{p_i}(x_2), f_{p_i}(x_3), \dots)$ determines a homeomorphism iff $m_\alpha(p_i) = \infty$. It follows that for any integer positive integer $a$ which is a product of primes with multiplicities in $\alpha$, $\hat{f}_a$ (as defined for $p_i$ ) is a homeomorphism. Let $\hat{f}_{\frac{a}{b}} = \hat{f}_a \circ \hat{f}_b^{-1}$. In the case when $\mathcal{K}_\alpha$ has two endpoints $g : (x_1, x_2, \dots) \longmapsto (-x_1, -x_2, \dots)$ is also a homeomomorphism. Homeomorphisms of the form $\hat{f}_{\frac{a}{b}}$ or of the form $g \circ \hat{f}_{\frac{a}{b}}$ will be referred to as the standard homeomorphisms on $\mathcal{K}_\alpha$.

In each isotopy class of $\mathcal{K}_\alpha$, there is exactly one of these standard homeomorphisms. Since isotopic homeomorphisms permute the composants the same way, they would have to leave the same number of composants invariant. Now we are ready to prove Theorem 3.6.4.

## 3.11 Number of Fixed Points of Homeomorphisms

For a given map $f : X \longrightarrow X$ on continuum $X$, let $FPS(f)$ denote the set of fixed points of $f$. The following is the proof of Theorem 3.6.4.

**Proof:** In view of theorem 3.6.4, we may assume $m_\alpha(3) = 0$. By Lemma 3.6.2 if $p \in \alpha$ then $\hat{f}_p\left(-\frac{1}{2}\right) = -\frac{1}{2}$. It follows therefore that

$$\xi = \left(-\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \dots\right) \in \mathcal{K}_\alpha.$$

Let $\Gamma_\xi$ be the composant of $\xi$. We shall show that every homeomorphism leaves this composant invariant. It's enough to show that the standard homeomorphisms leave this composant invariant.

Let $\hat{f}_{\frac{a}{b}}$ be a standard homeomorphism. Since $\xi$ is a fixed point for all Chebychev polynomials of prime degree (except the Chebychev polynomial of degree 3), $\xi$ is a fixed point of $\hat{f}_a$ and of $\hat{f}_b^{-1}$. Therefore $\xi$ is a fixed point for $\hat{f}_{\frac{a}{b}}$. Observe that the point $\hat{1} = (1, 1, 1, \dots)$ is another fixed point and the composant of $\hat{1}$ is different from $\Gamma_\xi$. The results then follow from Lemma 3.9.4. ∎

**Theorem 3.11.1** *Let $p$ be a prime and let $\mathcal{K}_p = \mathcal{K}_{(p,p,p,\dots)}$. Let $h : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ be homeomorphism. Then $h^{2n}$ has at least $p^{2n}$ fixed points.*

**Proof:** Observe that if $h$ is isotopic to either the identity or to the map $g : (x_1, x_2, \dots) \longmapsto (-x_1, -x_2, \dots)$, then $h$ has either uncountably many fixed points or uncountably many points of period 2. In either case the result is true. So assume without loss of generality that $h$ is isotopic to some power $k \geq 1$ of one of the standard homeomorphisms, i.e. either $\hat{f}_{p^k}$ or $g \circ \hat{f}_{p^k}$, in which case $h^{2n}$ is isotopic to $\hat{f}_{p^{2kn}}$. Since the Chebychev polynomial $f_{p^{2kn}}$ has $p^{2kn}$ fixed points and each one of them generates a fixed point of $\hat{f}_{p^{2kn}}$, the result follows by setting $k = 1$ for the minimum. ■

**Theorem 3.11.2** *Let $\alpha = (p_1, p_2, p_3, \dots)$ with $m_\alpha(p_k) < \infty$ for all $k$. Then every homeomorphism on $\mathcal{K}_\alpha$ has either uncountably many fixed points or uncountably many points of period 2.*

**Proof:** The isotopy class group of $\mathcal{K}_\alpha$ has only two elements, the identity and the map $g : (x_1, x_2, \dots) \longmapsto (-x_1, -x_2, \dots)$. The identity leaves uncountably many composants invariant and $g^2 =$ identity. ■

**Theorem 3.11.3** *Let $\alpha = (p_1, p_2, p_3, \dots)$ be a sequence of primes and let $h : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ be homeomorphism. Let $m$ be the number of elements in the set*

$$\bigcap_{\{p | m_\alpha(p) = \infty\}} FPS(f_p).$$

*Then the number of points in $FPS(h^2)$ is at least $m$.*

The number $m$ is obtained by solving a system of equations of the form $f_{p_i}(x) = x$, one for each prime that occurs infinitely often in $\alpha$. $m$ can also be obtained in the following way:

For a given prime $p$, the fixed points of $f_p$ are the real parts of the complex numbers satisfying either $z^p = z$ or $z^p = \bar{z}$. Equivalently $z^{p-1} = 1$ or $z^{p+1} = 1$.

Therefore, for a given prime $p$, the fixed points set is given by

$$FPS(f_p) = \left\{ \cos \frac{2\pi k}{p+1} : k = 1, 2, \dots \left\lceil \frac{p-1}{2} \right\rceil \right\} \bigcup \left\{ \cos \frac{2\pi k}{p-1} : k = 0, 1, \dots \left\lfloor \frac{p-1}{2} \right\rfloor \right\}.$$

For given primes $p$ and $q$, the cardinality of $FPS(f_p) \bigcap FPS(f_q)$ is equal to the cardinality of $F_p \cap F_q$ where for a given integer $n$ ,

$$F_n = \left\{ \frac{2k}{n+1} : k = 1, 2, \ldots \left\lceil \frac{n-1}{2} \right\rceil \right\} \bigcup \left\{ \frac{2k}{n-1} : k = 0, 1, \ldots \left\lfloor \frac{n-1}{2} \right\rfloor \right\},$$

where $\lfloor x \rfloor$ denotes the largest integer smaller than or equal to $x$ and $\lceil x \rceil$ is the smallest integer lager than or equal to $x$.

# CHAPTER 4
# HOMEOMORPHISMS OF KNASTER CONTINUA

The purpose of this chapter is twofold. The first aim is to catalogue results about the entropy of homeomorphisms on Knaster continua, the computation of which was done in Barge [4] and Kwapisz [17]. These computations were based on Bowen's theorem on the entropy of quotients. Our contribution is to show that Bowen's theorem does not generalize to non-compact spaces. We shall, to this end, give an example of a locally compact (but non compact) space on which the quotient rule does not apply.

Our second is goal to construct higher-dimensional Knaster continua. These will be constructed using toral homomorphisms induced by invertible matrices with integer entries. We show, for dimension two and above, that there are points in these continua that are homeomorphically distinguishable from other points in the spaces. We prove that for odd dimensions, a homeomorphism defined on these spaces lifts to two homeomorphisms on the solenoid. We shall also raise some questions for further research about possible generalization of results known about the one dimensional Knaster continua.

## 4.1 Entropy of Quotients.

Let $(X, d)$, $(Y, e)$ be metric spaces. Let $T : X \to X$ and $S : Y \to Y$ be maps. Let $\pi : X \to Y$ be a surjective map satisfying $\pi \circ T = S \circ \pi$. In [1], Bowen showed that if X and Y are compact then

$$h_d(T) \leq h_e(S) + \sup_{y \in Y} h_d(T, \pi^{-1}(y)).$$

A map such as $\pi$ is referred to as a semi-conjugacy. If $\pi$ is a homeomorphism, then it is referred to as a conjugacy. Conjugate maps have the same entropy and same period orbits and are dynamically equivalent. Each homeomorphism on the Knaster continuum is semi-conjugate to one on the solenoid.

The importance of the following example is to show that even though entropy can be computed by taking suprema of entropies computed on compact subsets, the estimate may fail if the space $Y$ is non compact. In this example, we have a uniformly continuous function $T$ with positive entropy yet

$$h_e(S) = \sup_{y \in Y} h_d(T, \pi^{-1}(y)) = 0.$$

Let $\Sigma_2 := \prod_{n=-\infty}^{\infty} \{0,1\}$, the space of bi-infinite sequences of 0's and 1's. A point $s \in \Sigma_2$ is represented as $s = (\ldots, s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n, \ldots)$ where each $s_n \in \{0,1\}$. We shall write $s = (\ldots s_{-n} \ldots s_{-1} : s_0 s_1 \ldots s_n \ldots)$ to the effect of separating the symbol sequence into two parts with both parts being infinite.

For our example let $X$ be the union of the spaces $X_1$ and $X_2$ described below.

$$X_1 := A \times \Sigma_2,$$

where $A := \mathbb{N} \times \{2^s | s \in \mathbb{Z}\}$, also

$$X_2 := B \times \{(\ldots 0 \ldots 0.00 \ldots 0 \ldots)\}$$

where $B = \mathbb{N} \times \{0\} = \bar{A} \setminus A$ with the operation of closure taken in $\mathbb{R}^2$. Generally, an element $x \in X$ will be of the form $(x_1, x_2, s)$ where $x_1, x_2 \in \mathbb{R}$ and $s \in \Sigma_2$.

For $x = (x_1, x_2, s)$ and $x' = (x_1', x_2', s') \in X$, define the distance between them to be

$$d(x, x') = \sup\{|x_1 - x_1'|, |x_2 - x_2'|, \sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0,t_1+t_2\}} s_i - 2^{\min\{0,t_1'+t_2'\}} s_i'\right|}{2^{|i|}}\}\}$$

where $t$, and $t'$ are such that $x_2 = 2^t$ and $x_2' = 2^{t'}$.

**Claim 1** *With the above, $X$ is a locally compact metric space.*

**Proof:** Clearly $d(x,x') \geq 0$ for all $x, x' \in X$ with equality if and only if $x = x'$. Also $d(x,x') = d(x',x)$ for all $x, x' \in X$. Now suppose $x = (x_1, x_2, s)$, $x' = (x_1', x_2', s')$, $x = (x_2'', x_2'', s'') \in X$, then

$$
\begin{aligned}
d(x,x') &= d((x_1,x_2,s),(x_1',x_2',s')) \\
&= \sup\left\{ |x_1 - x_1'|, |x_2 - x_2'|, \sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0,t_1+t_2\}}s_i - 2^{\min\{0,t_1'+t_2'\}}s_i'\right|}{2^{|i|}} \right\} \\
&\leq \sup\Bigg\{ |x_1 - x_1''| + |x_1'' - x_1'|, |x_2 - x_2''| + |x_2'' - x_2'|, \\
&\sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0,t_1+t_2\}}s_i - 2^{\min\{0,t_1''+t_2''\}}s_i'' + 2^{\min\{0,t_1''+t_2''\}}s_i'' - 2^{\min\{0,t_1'+t_2'\}}s_i'\right|}{2^{|i|}} \Bigg\} \\
&\leq \sup\Bigg\{ |x_1 - x_1''| + |x_1'' - x_1'|, |x_2 - x_2''| + |x_2'' - x_2'|, \\
&\sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0,t_1+t_2\}}s_i - 2^{\min\{0,t_1''+t_2''\}}s_i''\right| + \left|2^{\min\{0,t_1''+t_2''\}}s_i'' - 2^{\min\{0,t_1'+t_2'\}}s_i'\right|}{2^{|i|}} \Bigg\} \\
&\leq \sup\Bigg\{ |x_1 - x_1''| + |x_1'' - x_1'|, |x_2 - x_2''| + |x_2'' - x_2'|, \\
&\sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0,t_1+t_2\}}s_i - 2^{\min\{0,t_1''+t_2''\}}s_i''\right|}{2^{|i|}} + \sum_{i=-\infty}^{\infty} \frac{\left|2^{\min\{0,t_1''+t_2''\}}s_i'' - 2^{\min\{0,t_1'+t_2'\}}s_i'\right|}{2^{|i|}} \Bigg\} \\
&= d(x,x'') + d(x'',x').
\end{aligned}
\tag{4.1}
$$

Next we prove that $(X,d)$ is locally compact. Let $\epsilon > 0$ be given (we may assume $0 < \epsilon < \frac{1}{2}$). Consider an $\epsilon$–neighborhood $U$ of $(0,0,\bar{0})$, where $\bar{0} = \{\ldots 0 \ldots 0.00 \ldots 0 \ldots\}$. Let $N$ be the smallest integer such that $2^N \geq \epsilon$ .Then the subspace $V = \{(0,2^j,s) | j < N, s \in \Sigma_2\} \subset U$ and $\bar{V}$ is compact. Observe that it's enough to check for such a neighborhood. ∎

Let $Y = \bar{A}$ be equipped with the Euclidean metric from $\mathbb{R}^2$. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift map defined by

$$\sigma(\{\ldots, s_{-n}, \ldots, s_{-1}, s_0, s_1, \ldots, s_n, \ldots\}) = \{\ldots s_{-n} \ldots s_{-1} s_0 . s_1 \ldots s_n \ldots\}$$

and let $\tau : \Sigma_2 \to \Sigma_2$ be the identity map on $\Sigma_2$. Define $T : X \to X$ by

$$T(x_1, x_2 = 2^t, s) = (x_1 + 1, x_2, \sigma(s)) \text{ if } x_1 \le t$$

$$T(x_1, x_2 = 2^t, s) = (x_1 + 1, x_2, \tau(s)) \text{ if } x_1 > t$$

and $T(x, 0, \bar{0}) = (x + 1, 0, \bar{0})$

**Claim 2** *$T$ is uniformly continuous on $X$*

**Proof:** This follows from the fact that $T\left(B_{\frac{\epsilon}{2}}(x)\right) \subset B_\epsilon(T(x))$, where $B_a(z)$ stands for the ball of radius $a$ around the point z. ■

**Theorem 4.1.1** *$T$ has positive topological entropy.*

**Proof:** We need to find a compact set $K$ such that $h_d(T, K) > 0$. Let $K$ be the closure of $\left\{(x_1, x_2, s) \in X | x_1 = 0, x_2 \le \frac{1}{2}\right\} \subseteq cl_X(B_{\frac{1}{2}}(0))$. For any given $\epsilon > 0$ let $m$ be the largest integer such that $0 < \epsilon \le \frac{1}{2^m}$. Then

$$r_0(\epsilon, K) \ge 2^m + 2^{m-1} + \cdots + 2 + 1 + 1$$

$$r_1(\epsilon, K) \ge 2^m + 2^m + 2^{m-1} + \cdots + 2^2 + 2 + 2 \ (m\text{+2 terms})$$

$$r_2(\epsilon, K) \ge 2^m + 2^m + 2^m + 2^{m-1} + \cdots + 2^2 + 2 + 2$$

$$\cdots$$

$$r_2(\epsilon, K) \ge (m + 1)2^m + 2^n$$

So that $\sup \frac{1}{n} \log r_n(\epsilon, K) \ge \sup \frac{1}{n} \log(m2^m + 2^n) \ge \sup \frac{1}{n} \log 2^n = \log 2$. Therefore $h_d(T, K) = \lim_{n\to\infty} \sup \frac{1}{n} \log r_n(\epsilon, K) \ge \log 2$. ■

Let $Y = \bar{X}_1$ and $\pi$ be the corresponding quotient map from $X$ to $Y$. $S : Y \to Y$ be the map $S(y_1, y_2) = (y_1 + 1, y_2)$.

**Claim 3** *$S$ has topological entropy zero.*

**Proof:** For each compact subset K of X and $\epsilon > 0$, the number $r_n(\epsilon, K)$ is constant for all $n$. It follows that $\lim_{n\to\infty} \sup \frac{1}{n} \log r_n(\epsilon, K) = 0$ for every $\epsilon > 0$ and therefore the topological entropy of S is 0. ∎

**Theorem 4.1.2** *For every $y \in Y.$, $h_d(T, \pi^{-1}(y)) = 0$.*

**Proof:** Let $y = (y_1, y_2) \in Y$, $y_1 = t_1, y_2 = 2^{t_2}$, $t_1, t_2 \in \mathbb{Z}$. So for any $x, x' \in \pi^{-1}(y)$,

$$d(x, x') = 2^{\min\{0, t_1+t_2\}} \sum_{i=-\infty}^{\infty} \frac{|s_i - s_i'|}{2^{|i|}}$$

where $x = (t_1, 2^{t_2}, s)$, $x' = (t_1, 2^{t_2}, s')$.

Since $T(x_1, 2^t, s) = (x_1 + 1, 2^t, s)$ if $x_1 > t$ and $T(x_1, 2^t, s) = (x_1 + 1, 2^t, \sigma(s))$ if $x_1 \leq t$, there exists an integer $m$ such that $T^j((x_1, 2^t, s)) = (x_1 + j, 2^t, \sigma^m(s))$ for all $j \geq m$. It follows that if $K$ is a compact subset of $\pi^{-1}(y)$, for a given $\epsilon > 0$, $r_n(\epsilon, K)$ attains a maximum (as $n$ increases) of $r_m(\epsilon, K)$. So $\lim_{n\to\infty} \sup \frac{1}{n} \log r_n(\epsilon, K) = \lim_{n\to\infty} \sup \frac{1}{n} \log r_m(\epsilon, K) = 0$ and the result follows. ∎

## 4.2 Entropy of a Homeomorphism of the Knaster Continuum

The proof of the following theorem may be found in Barge [4].

**Theorem 4.2.1** *Let $g : \mathcal{K}_\alpha \longrightarrow \mathcal{K}_\alpha$ be a homeomorphism, $\alpha = (p_1, p_2, \dots)$ then there exists positive integers $a$ and $b$ both products of primes with infinite multiplicities in $\alpha$, and $h_d(g) = \max\{\log a, \log b\}$.*

It is further shown that isotopic homeomorphisms on $\mathcal{K}_\alpha$ have the same entropy. Therefore, in a sense, the standard homeomorphisms are representatives of all homeomorphism on $\mathcal{K}_\alpha$ both dynamically and homotopically.

## 4.3 Generalizing Knaster Continua

We generalize the definition of Knaster continua to higher-dimensions and characterize analogues of "end points" to these dimensions. These points turn out

to be vertices of immersed infinite cones over real projective spaces and are homologically distinguishable in dimension higher than 2. We determine the number of such points that a specified Knaster continua may have. Finally, we show that a homeomorphism on any such continuum lifts to exactly to homeomorphisms on the solenoid corresponding to that continuum. In the one dimensional case, the number of 2's in the sequence of primes for a given Knaster continuum determines the number of endpoints. In case when the defining matrices are all diagonal with prime numbers as entries, the number of 2's on the leading diagonals of these matrices determine the number of homologically distinguishable points in the higher-dimensional Knaster continua.

Using these points we show that a homeomorphism on the so defined continuum lifts to two homeomorphisms on the solenoid. Let $S^1$ be the unit circle in the complex plane. Let $p : \mathbb{R} \longrightarrow S^1$ be the quotient map given by $t \longmapsto \exp(2\pi i t)$. Let $\mathbb{T}^n = \prod_{i=1}^{n} S^1$ be the $n$-torus with the product topology from $S^1$. Let $p_n : \mathbb{R}^n \longrightarrow \mathbb{T}^n$ be the covering map that is the product of $p$ i.e.

$$p_n(t_1, t_2, \ldots, t_n) = (p(t_1), p(t_2), \ldots p(t_n)).$$

Observe that if $f : \mathbb{T}^n \longrightarrow \mathbb{T}^n$ is a homomorphism, there is a unique homomorphism on $\mathbb{R}^n$ represented by an $n \times n$ matrix $M$ with integer entries satisfying $f \circ p_n = p_n \circ M$.

## 4.4 Weak Solenoids

Let $\mu = (M_1, M_2, \ldots)$ be an infinite sequence of $n \times n$ nonsingular matrices with integer coefficients. Define the $\mu$-adic solenoid $\mathcal{S}_\mu$ to be the inverse limit space of mappings $f_{M_j}$, on the torus $\mathbb{T}^n$. We write

$$\mathcal{S}_\mu = \varprojlim \left(\mathbb{T}^n, f_{M_j}\right) = \{(x_k)_{k=1}^{\infty} : x_k = f_{M_k}(x_{k+1}), k \geq 1\}$$

where $f_{M_j}$ are maps induced on $\mathbb{T}^n$ by the linear transformations on $\mathbb{R}^n$ whose matrices are $\{M_j\}_{j=1}^{\infty}$.

$$\begin{array}{ccccccc}
R^n & \xleftarrow{M_1} & R^n & \xleftarrow{M_2} & R^n & \xleftarrow{M_3} & \cdots \\
{\scriptstyle p_n}\downarrow & & {\scriptstyle p_n}\downarrow & & {\scriptstyle p_n}\downarrow & & \\
T^n & \xleftarrow{f_{M_1}} & T^n & \xleftarrow{f_{M_2}} & T^n & \xleftarrow{f_{M_3}} & \cdots
\end{array}$$

Equipped by the relativitized product topology from $\prod_{i=1}^{\infty} \mathbb{T}^n$, $\mathcal{S}_\Lambda$ is a connected compact metric topological group. In the case when $n = 1$, we get the well known 1-dimensional solenoids. Let $\bar{p}_n : \mathbb{R}^n \longrightarrow \mathcal{S}_\mu$ be the map induced by the projections $p_n : \mathbb{R}^n \longrightarrow \mathbb{T}^n$, i.e

$$\bar{p}_n\left(\underline{t}\right) = \left(p\left(\left(\prod_{i=0}^{k-1} M_i\right)^{-1} \cdot \underline{t}\right)\right)_{k=1}^{\infty},$$

where $\bar{t}$ is the transpose of the vector $(t_1, t_2, \ldots, t_n)$. We denote the image of $\bar{p}_n$ by $\Gamma_n$ and we may drop the indices if the dimension $n$ is understood. All other arc components in $\mathcal{S}_\mu$ are cosets of $\mathcal{S}_\mu$ by the subgroup $\Gamma_n$. We shall denote the composant of an element $x \in \mathcal{S}_\mu$ by $\Gamma(x)$. The sequence $\mu$ can be chosen so that $\bar{p}_n$ is a 1-1. Then, $\bar{p}_n$ is continuous homomorphism of $\mathbb{R}^n$ onto a dense subgroup $\Gamma_\mu$ of $\mathcal{S}_\mu$ which is the arc component of the identity element in $\mathcal{S}_\mu$ [20, section 5].

The subgroup $\Lambda_\mu := \{z = (z_k)_{k=0}^{\infty} \in \mathcal{S}_\mu : z_0 = \underline{1}\}$ of $\mathcal{S}_\mu$ is homeomorphic to the Cantor set [20, lemma 5.1]. It follows from McCord [20, Lemma 5.4] that $\bar{\pi} : \Lambda_\mu \times \mathbb{R}^n \longrightarrow \mathcal{S}_\mu$ defined by $\bar{\pi}(w, t) = \bar{p}_n(\underline{t}) \cdot w$ is a local homeomorphism of $\Lambda_\mu \times \mathbb{R}^n$ onto $\mathcal{S}_\mu$. Let $d_{\Lambda_\mu}$ be an invariant metric on $\Lambda_\mu$ [13]. This induces invariant metrics on $\Lambda_\mu \times \mathbb{R}^n$ and $\mathcal{S}_\mu$ given by

$$d_{\Lambda_\mu \times \mathbb{R}^n}\left((w,t), (\tilde{w}, \tilde{t})\right) := \max\left\{d_{\Lambda_\mu}(w, \tilde{w}), \left|\left|t - \tilde{t}\right|\right|\right\} \tag{4.2}$$

and

$$d_{\mathcal{S}_\mu}(z, \tilde{z}) := \min\left\{d_{\Lambda_\mu \times \mathbb{R}^n}\left((w,t), (\tilde{w}, \tilde{t})\right) : \bar{\pi}(w,t) = z,\ \bar{\pi}\left(\tilde{w}, \tilde{t}\right) = \tilde{z}\right\} \tag{4.3}$$

where $||\cdot||$ is the Euclidean norm on $\mathbb{R}^n$. These metrics make $\pi$ into a local isometry.

## 4.5 Knaster Continua from Weak Solenoids

Given a sequence

$\mu = (M_1, M_2, \dots)$ of $n \times n$ invertible matrices with integer entries, define the $\mu$-adic Knaster continuum $\mathcal{K}_\mu$ to be the continuum obtained by identifying the element $x \in \mathcal{S}_\mu$ with it's inverse. So we have map from $\mathcal{S}_\mu$ to $\mathcal{K}_\mu$ that is at most 2-1. Let

$$\pi : \mathcal{S}_\mu \to \mathcal{K}_\mu$$

be the quotient map. Define the metric $d_{\mathcal{K}_\mu}$ by the following.

$$d_{\mathcal{K}_\mu}([z], [\tilde{z}]) = \min\left\{d_{\mathcal{S}_\mu}(z, \tilde{z}), d_{\mathcal{S}_\mu}(z^{-1}, \tilde{z})\right\} \tag{4.4}$$

## 4.6 Distinguishable Points in Knaster Continua

Let $\mu = (M_1, M_2, \dots)$ be a sequence of diagonal matrices with integer entries and let $\mathcal{K}_\mu$ be the $\mu$-adic Knaster continuum. For our purpose, an element $x$ of a group $G$ will be referred to as a *binod* element if $x^2 = e$. In $\mathcal{S}_\mu$, each of the equivalence classes in $K_\mu$ of such elements has a single element in them. These elements come from binod elements in the torus $\mathbb{T}^n$; There are $2^n$ such elements in the torus:

$$(1, 1, 1, \dots, 1), (-1, 1, 1, \dots, 1), (1, -1, 1, \dots, 1), \dots, (-1, -1, \dots, -1).$$

and these are covered by the $2^n$ elements:

$$(0, 0, 0, \dots, 0), (\frac{1}{2}, 0, 0, \dots, 0), (0, \frac{1}{2}, 0, \dots, 0), \dots, (\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}) \in \mathbb{R}^n.$$

**Theorem 4.6.1** *Let*

$$M = \begin{pmatrix} p_1 & 0 & \dots & 0 \\ 0 & p_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & p_n \end{pmatrix}$$

*and let $\mathcal{S}_M = \mathcal{S}_\mu$, with $\mu = (M, M, \dots)$, then the number of binod elements in $\mathcal{S}_\mu$ is $2^r$ where $r$ is the number of primes on the diagonal of $M$ that are distinct from 2.*

Each of the $2^r$ binod elements in the solenoid leads to such a point in the Knaster continuum. We shall use homology to distinguish these points from the rest of the points in $\mathcal{K}_\mu$ for dimension $n \geq 3$. We observe that this analysis does not work in dimension 2.

Observe that arc components in the solenoid are immersions of $\mathbb{R}^n$ in $\mathcal{S}_\mu$ and if a point $x \in \mathcal{S}_\mu$ is a binod element, then all points in its arc component have their inverses in that arc component. This follows from the continuity of the map $\psi : w \mapsto w^{-1}$ on $\mathcal{S}_\mu$. It follows that the binod elements are the images of the origin under the map $\phi_x \circ \bar{p}$ that maps $\mathbb{R}^n$ to the arc component $\Gamma(x)$ of $x$ in the solenoid. Here $\phi_x := w \mapsto x \cdot w$.

Observe that inverses in $\mathbb{R}^n$ lie on spheres as antipodal points. So identifying inverses in the solenoid corresponds to identifying antipodal points of spheres centered at the origin in $\mathbb{R}^n$. Each of these spheres then becomes an $(n-1)$-dimensional projective plane. The resulting space is an infinite cone over projective space with the binod elements as the the vertices.

Let $\mathbb{P}^n$ be the space obtained from $\mathbb{R}^n$ by identifying $\bar{t}$ with $-\bar{t}$ and let $p : \mathbb{R}^n \to \mathbb{P}^n$ be the quotient map. For a point $z \in \mathcal{K}_\mu$ let $C(z)$ be the arc component of $z$ in $\mathcal{K}_\mu$. We note that if $z \in \mathcal{S}_\mu$ is a binod element, then $C(z)$ is obtained from $\Gamma(z)$ by identifying points and their inverses.

**Theorem 4.6.2** *Consider $\bar{p} : \mathbb{R}^n \to \Gamma$, (as earlier defined). For a given binod element $z \in \mathcal{S}_\mu$ define $\phi_z : \mathcal{S}_\mu \mapsto \mathcal{S}_\mu$ by $\phi_z(w) = w \cdot z$. The the following diagram*

$$\begin{array}{ccc} \mathbb{R}^n & \xrightarrow{\phi_z \circ \bar{p}} & \Gamma(z) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi|_{\Gamma(z)}} \\ \mathbb{P} & \xrightarrow{\Psi} & C(z) \end{array}$$

*commutes.*

The following lemmas follow from the homology exact sequences of pairs [27]

**Lemma 4.6.3** *Let $z$ be a binod element in $\mathcal{S}_\mu$, $\mu$ a sequence of diagonal matrices with integer entries. Let $C(z)$ be the arc component of $[z] \in \mathcal{K}_\mu$. Then*

$$H_n(C(z), C(z) - z) = H_n(\mathbb{R}P^{n-1}).$$

*where $H_n(X, A)$ is the homology of the space $X$ relative to a subset $A$.*

**Lemma 4.6.4** *Let $z$ be a binod element in $\mathcal{S}_\mu$, $\mu$ a sequence of diagonal matrices with integer entries. Let $C(z)$ be the arc component of $[z] \in \mathcal{K}_\mu$. Suppose $x \in C(z)$, $x \neq z$, then*

$$H_n(C(z), C(z) - x) = H_n(S^{n-1}).$$

**Lemma 4.6.5** *Let $z$ be an element in $\mathcal{S}_\mu$, $z$ not a binod element in $\mathcal{S}_\mu$, $\mu$ a sequence of diagonal matrices with integer entries. Let $C(z)$ be the arc component of $[z] \in \mathcal{K}_\mu$. Suppose $x \in C(z)$, then*

$$H_n(C(z), C(z) - x) = H_n(S^{n-1}).$$

## 4.7 Lifting Homeomorphisms from Knaster Continua to Solenoids

**Lemma 4.7.1** *Let $\tau : \mathbb{R}^n \to \mathbb{R}^n$ ($n$ an odd positive integer) be a homeomorphism that fixes the origin and is isotopic to the identity. Let $\rho > \epsilon > 0$ be positive numbers, then there exists a $t$ with $\|t\|, \|\tau(t)\| \in [0, \rho]$ such that $\|t + \tau(t)\| = \epsilon$.*

**Proof:** Consider a sphere $S_\epsilon$ centered at the origin and of radius $\epsilon$. Suppose for every $t$ on $S_\epsilon$ the angle between the vector $t$ and $\tau(t)$ is bigger than $\frac{\pi}{2}$. Define $g : S^n \to S^n$ by

$$g(x) = \frac{\tau(\epsilon x)}{\|\tau(\epsilon x)\|}$$

and let $f : S^n \to S^n$ be the antipodal map $x \mapsto -x$. Then $g$ is homotopic to $f$ for if $H : \mathbb{R}^n \times I \to \mathbb{R}$ is the isotopy between $\tau$ and the identity map, then

$$\tilde{H}(x, t) := \frac{H(\epsilon x, t)}{\|H(\epsilon x, t)\|}$$

will be a homotopy between the identity and $g$. This is a contradiction since $S^n$ is an even dimensional sphere. The identity is not homotopic to the antipodal map in even dimension.

So there is $t$ on $S_\epsilon$ such that the angle between $t$ and $\tau(t)$ is less than or equal to $\frac{\pi}{2}$ and therefore $\|t + \tau(t)\| > \epsilon$ By continuity of $t + \tau(t)$ and $\tau(0) = 0$, there is a $t$ such that $\|t + \tau(t)\| = \epsilon$. ■

**Lemma 4.7.2** *There is a $\rho > 0$ such that if $\epsilon < \rho$, and $\tau : \mathbb{R}^n \to \mathbb{R}^n$ ($n$ an odd positive integer) is a homeomorphism isotopic to the identity and satisfying*

$$d_{\mathcal{K}_\mu}(\pi(s(t)z), \pi(s(-\tau(t))z) < \epsilon, t \in \mathbb{R}^n,$$

*then*

$$d_{\mathcal{S}_\mu}(z, z^{-1}) < 3\epsilon.$$

**Proof:** Since the projection $\Lambda \times \mathbb{R}^n \to \mathcal{S}_\mu$ is a local isometry, there exists a $\rho > 0$ such that $d_{\mathcal{S}_\mu}(s(t)z, s(-\tau(t))z) = \|t + \tau(t)\|$ for $t$ satisfying $\|t\|, \|\tau(t)\| \in [0, \rho]$. By hypothesis

$$\begin{aligned} &d_{\mathcal{K}_\mu}(\pi(s(t)z), \pi(s(-\tau(t))z)) \\ &\quad = \min\left\{d_{\mathcal{S}_\mu}(s(t)z, s(-\tau(t))z), d_{\mathcal{S}_\mu}(s(t)z, s(+\tau(t))z^{-1})\right\} < \epsilon < \rho. \end{aligned} \tag{4.5}$$

It follows from $\tau$ being isotopic to the identity (lemma 4.7.1) that there is a $t$ satisfying $\|t\|, \|\tau(t)\| \in [0, \rho]$, $\|t - \tau(t)\| < \epsilon$ and $\|t + \tau(t)\| = \epsilon$ ($t$ is chosen so that the angle between $t$ and $\tau(t)$ is less than $\pi/2$).

It follows from (4.4) for that specific $t$ that $d_{\mathcal{S}_\mu}(s(t)z, s(\tau(t))z^{-1}) < \epsilon$ and therefore

$$\begin{aligned} &d_{\mathcal{S}_\mu}(z, z^{-1}) \\ &\quad \le d_{\mathcal{S}_\mu}(z, s(t)z) + d_{\mathcal{S}_\mu}(s(t)z, s(\tau(t)z^{-1})) + d_{\mathcal{S}_\mu}(s(\tau(t)z^{-1}), z^{-1}) \\ &\quad < \|t\| + \epsilon + \|\tau(t)\| < 3\epsilon. \end{aligned}$$

Therefore the proof is complete. ∎

**Theorem 4.7.3** *Let $\mu$ be a sequence of $n \times n$ invertible matrices with integer entries, $n$ an odd positive integer, and assume the induced homomorphism $\bar{p} : \mathbb{R}^n \to \Gamma$ is $1-1$. Let*

$$h : \mathcal{K}_\mu \to \mathcal{K}_\mu$$

*be a homeomorphism that fixes any of the vertex elements. Then there exist exactly two distinct homeomorphisms $\tilde{h}_1,\ \tilde{h}_2 : \mathcal{S}_\mu \to \mathcal{S}_\mu$ such that $h_i \circ \pi = \pi \circ h,\ i = 1, 2$*

**Proof:** Let $\mathbb{P}^n = \mathbb{R}^n / \sim$ where $t \sim t'$ iff $t' = t^{-1}$ and let $k : \mathbb{P}^n \to \mathcal{K}_\mu$ be the immersion of $\mathbb{P}^n$ into $\mathcal{K}_\mu$. Let $\bar{h} := k^{-1} \circ h \circ k$. Since $n \geq 3$, $\tilde{\mathbb{R}}^n = \mathbb{R}^n \setminus \{0\}$ is a universal cover for $\tilde{\mathbb{P}}^n = \mathbb{P}^n \setminus \{k(0)\}$ and therefore there is a homeomorphism $\bar{\bar{h}} : \mathbb{R}^n \to \mathbb{R}^n$ which lifts $\bar{h}$ (first lift to $\tilde{\mathbb{R}}^n$ and then extend to the origin). Define $\tilde{h} : \Gamma \to \Gamma$ by $\tilde{h} := s \circ \bar{\bar{h}} \circ s^{-1}$. We shall show that $\tilde{h}$ is uniformly continuous.

$$\begin{tikzcd}
R^n \arrow[rrr, "\bar{\bar{h}}"] \arrow[dr, "s"] \arrow[ddd, "p"'] & & & R^n \arrow[dl, "s"'] \arrow[ddd, "p"] \\
& S^n \arrow[r, "\tilde{h}"] \arrow[d, "\pi"'] & S^n \arrow[d, "\pi"] & \\
& K^n \arrow[r, "h"] & K^n & \\
P^n \arrow[ur, "k"] \arrow[rrr, "\bar{h}"'] & & & P^n \arrow[ul, "k"']
\end{tikzcd}$$

Fix $\epsilon > 0$ such that $\frac{\epsilon}{3} < \rho$, there is a $\delta > 0$ such that $d_{\mathcal{K}_\mu}(h(x), h(y)) < \frac{\epsilon}{8}$ whenever $d_{\mathcal{K}_\mu}(x, y) < \delta,\ x, y \in \mathcal{K}_\mu$. Suppose that $d_{\mathcal{S}_\mu}(s(x_1), s(x_2)) < \delta$. We shall show that $d_{\mathcal{S}_\mu}(\tilde{h}(s(x_1)), \tilde{h}(s(x_2))) < \epsilon$. It follows from translations being isometries of $\mathcal{S}_\mu$, that $d_{\mathcal{S}_\mu}(s(x_1 + t), s(x_2 + t)) < \delta$ for all $t$. So $h \circ k\,([x_i + t]) = k\,([y_i + \tau_i\,(t)]),\ y_i = \bar{\bar{h}}(x_i)$, where $\tau_i : \mathbb{R}^n \to \mathbb{R}^n$ homeomorphisms of $\mathbb{R}^n fori = 1, 2$. By the choice of $\delta$,

$$d_{\mathcal{S}_\mu}(k([y_1 + \tau_1(t)])), k([y_2 + \tau_2(t)]) < \frac{\epsilon}{8}, \forall t. \tag{4.6}$$

If $d_{\mathcal{S}_\mu}(s(y_1), s(y_2)) < \frac{\epsilon}{8}$ then the proof is complete. Otherwise $d_{\mathcal{S}_\mu}(s(y_1), s(-y_2)) < \frac{\epsilon}{8}$. It follows that $d_{\mathcal{S}_\mu}(s(y_1 + t), s(-y_2 + t)) < \frac{\epsilon}{8}$ for all $t \in \mathbb{R}^n$. This implies that $d_{\mathcal{K}_\mu}(\pi(s(y_1 + t)), \pi(s(-y_2 + t))) < \frac{\epsilon}{8}$ for all $t \in \mathbb{R}^n$. Let $\tau := \tau_2 \circ \tau_1^{-1}$. By the triangle inequality,

$$\begin{aligned}
&d_{\mathcal{K}_\mu}(\pi(s(y_2 + \tau(t))), \pi(s(-y_2 + t))) \\
&\quad \leq d_{\mathcal{K}_\mu}(\pi(s(y_2 + \tau(t))), \pi(s(y_1 + t))) + d_{\mathcal{K}_\mu}(\pi(s(y_1 + t)), \pi(s(-y_2 + t))) \\
&\quad \leq \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4}.
\end{aligned}$$

In addition $\tau_i(t) = h(x_i + t) - h(x) fori = 1, 2$, so $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ defined by $H(t, s) := h(s(x_2 - x_1) + x_1 + t) - h(s(x_2 - x_1) + x_1)$ is an isotopy between $\tau_1$ and $\tau_2$. It follows that $\tau$ is isotopic to the identity.

It follows from lemma 4.7.2 that $d_{\mathcal{S}_\mu}(s(y_2), s(-y_2)) < \frac{3\epsilon}{4}$ and therefore that $d_{\mathcal{S}_\mu}(s(y_1), s(y_2)) \leq d_{\mathcal{S}_\mu}(s(y_1), s(-y_2)) + d_{\mathcal{S}_\mu}(s(y_{-2}), s(y_2)) < \frac{\epsilon}{8} + \frac{3\epsilon}{4}$. ∎

CHAPTER 5
CONCLUSION

## 5.1 Summary

We have constructed spaces that we have called higher-dimensional Knaster continua. Our construction has been based on the definitions of one dimensional Knaster continua. We have shown that each of such continua (with the exception of dimension 2) has a finite number of distinguishable points. We have homology theory to show that these points are preserved under homeomorphisms.

We have shown that homeomorphisms defined on the higher-dimensional Knaster continua lift to homeomorphism of the generalized solenoid. The importance of this is in knowing that the generalized solenoid has such nice properties as of being a topological group and homogeneous. For instance every map on a topological group that fixes the identity element is homotopic to a homomorphism. If a homeomorphism $\tilde{h}$ of the solenoid is a lift of a homeomorphism $h$ of the Knaster continuum, then some iterate of $\tilde{h}$ fixes the identity element in the generalized solenoid. It follows that the specific itirate of $\tilde{h}$ is homotopic to a homorphism. In addition the topological entropy of the homeorphisms of these higher-dimensional Knaster continua can be estimated using Bowen's results [8]. In relation to these results, we have shown that Bowen's theorem [8] does not generalize to locally compact spaces that are not compact.

Results of chapter of this dissertation concern fixed point theory of one dimensional Knaster continua. Aarts and Fokkink [2] proved that a homeomorphism of the Knaster bucket handle must have two fixed points. Our main contribution has been to straighten out an erroneous generalization of their result [2, theorem 15]. We have

proved the correct versions of that generalization. In particular we have shown that the number of fixed points a general homeomorphism on such continua may have, depends on the whether 2 appears in the defining sequence infinitely often.

Looking back at the theory of these spaces in one dimension, we are motivated to ask the questions in the next section.

## 5.2 Questions

1. There does not seem to be an obvious way to extend theorem 4.7.3 to the two dimensional Knaster continua. One of the difficulties is the lack of identifiable points in dimension 2. This leads to the *question*: Is this theorem true in dimension 2 or in any other even dimension?

2. In dimension one, every homeomorphism of the Knaster Continuum is isotopic to a power of a shift map, in other words, the *mapping class group* of such continua is generated by the shift homeomomorphisms. *Question*: Find the *mapping class groups* of these higher-dimensional Knaster continua?

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